JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 6, November–December 2006

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Probabilistic Robust Linear Parameter-Varying Control of an F-16 Aircraft

Bei Lu*
California State University, Long Beach, California 90840
and
Fen Wu†
North Carolina State University, Raleigh, North Carolina 27695

DOI: 10.2514/1.22495

I. Introduction

OPERATIONAL capability at high angles of attack, especially near and at post stall regimes, is critical for next generation fighter aircrafts and uninhabited aerial vehicles [1]. However, significantly large levels of modeling uncertainty are inevitably encountered in flight control design for those regimes. The sources of uncertainty include variations in mass, inertia, and center of gravity positions, uncertainty in the aerodynamic data, etc. [2]. The maneuverability at high angles of attack poses a challenging control problem that requires guaranteeing both robust stability and robust performance in the presence of large parameter variations.

Traditional robust control techniques, like $\mathcal{H}_\infty$ and μ-synthesis, have been proven to be capable of producing robust uncertainty-tolerant controllers for next generation aircrafts [2,3]. However, those techniques focus on deterministic worst-case robust analysis and synthesis, which often lead to overly conservative stability bound estimate and high control effort. Moreover, a large number of conventional deterministic problems in robustness analysis and synthesis are shown to be NP-hard. To reduce conservatism and computational complexity, one approach is to shift the meaning of robustness from its usual deterministic sense to a probabilistic one [4]. In contrast to traditional robust control techniques, only a probabilistic solution is given, and a certain risk-level should be accepted. However, such a system may be viewed as being practically robust from an engineering point of view.

Algorithms derived in the probabilistic context are based on uncertainty randomization and usually called randomized algorithms, which may be divided into two families: methods based on statistical learning theory [5], and sequential methods based on subgradient iterations [6–8] or ellipsoid iterations [9,10]. The former can deal with nonconvex synthesis problems; however, it resorts to randomized search over the controller parameters to find a candidate solution. On the other hand, the sequential methods are formulated based on convex problems, thus avoiding the controller randomization issue [4].

The probabilistic robust control approach is still in the stage of algorithm development and improvement, and has not been explored in depth for flight control. The number of implementation of probabilistic techniques is therefore rather restricted. In the late 90s, Marrison and Stengel designed a linear quadratic regulator to control the nonlinear longitudinal dynamics of a hypersonic aircraft [11]. Recently, Wang and Stengel designed a robust flight control system for the high-incidence research model problem by combining stochastic robustness with nonlinear dynamic inversion [12]. Their work was based on statistical learning theory, and controllers were searched by using generic algorithms to minimize stochastic robustness cost functions. In our earlier paper, we applied an ellipsoid algorithm to design an $\mathcal{H}_\infty$ controller for a linearized F-16 longitudinal model [13]. Good stability and performance robustness have been achieved at the chosen flight condition.

The motivation for this research is twofold. First, the probabilistic control design method for linear time-invariant plants in [13] is generalized to linear parameter-varying (LPV) systems. This generalization is very important because of the relevance of LPV systems to nonlinear systems. The LPV control synthesis condition is known to be formulated as a convex problem with a set of parameter-dependent linear matrix inequalities (LMIs) [14–16]. Second, the current state of the art does not allow accurate aerodynamic modeling in the high angle of attack region. Because of its random nature, uncertainty in the aerodynamic data can be characterized using a statistical model, which can be handled effectively by the promising probabilistic robust control approach. Note that the study in this note focuses on the robustness issue with respect to the aerodynamic uncertainty at high angles of attack, and the results would be easily generalized to other parametric uncertainties, such as variations in mass and inertial properties.

Because of the convex formulation of LPV control synthesis, the sequential method is more suitable for dealing with uncertainties and designing probabilistic robust LPV controllers. An ellipsoid algorithm with a stopping rule proposed by Oishi [10] is used to determine feasible solutions to LMI synthesis conditions. The paper is organized as follows. In Sec. II, the ellipsoid algorithm is presented, which either gives a probabilistic solution with high confidence or detects that there is no deterministic solution in an approximated sense. Section III first provides a brief overview of robust control problem of an uncertain LPV system, and then discusses the computational issues when the algorithm is applied to the robust LPV control problem. In Sec. IV, a robust LPV controller is designed for an F-16 aircraft with large aerodynamic uncertainty, and the robust performance is tested through nonlinear simulations. Finally, the paper concludes with a summary in Sec. V.

II. Ellipsoid Algorithm with a Stopping Rule

In this section, the ellipsoid algorithm with a stopping rule is presented to determine feasible solutions to robust LMI constraints, which arise naturally from standard robust control problems when uncertainty is present in data matrices. State-space matrices can be affected by parametric and nonparametric uncertainties [7]. In the former case, entries of state matrices are functions of uncertain

---

Presented as Paper 6080 at the AIAA Guidance, Navigation, and Control, San Francisco, 15–18 August 2005; received 16 January 2006; revision received 29 June 2006; accepted for publication 21 August 2006.

*Assistant Professor, Department of Mechanical and Aerospace Engineering; blu3@csulb.edu. Member AIAA.

†Associate Professor, Department of Mechanical and Aerospace Engineering; fwu@eos.ncsu.edu. Member AIAA.

parameters which are bounded within intervals. In the latter case, the uncertainty is a norm-bounded matrix perturbing the nominal state matrices.

Assume that the uncertainty Δ is a random bounded matrix with a given probability density function f_Δ over its support set $\boldsymbol{\Delta}$. A robust LMI is defined as the matrix inequality [6,10]

$$F(x, \Delta) = F_0(\Delta) + \sum_{i=1}^{n} x_i F_i(\Delta) < 0, \qquad \forall \Delta \in \boldsymbol{\Delta} \tag{1}$$

where $x \in \boldsymbol{R}^n$ is the LMI optimization variable, and $F_i (i = 0, \ldots, n)$ are symmetric matrices. Generally, the robust LMI (1) needs to be checked for all values of Δ in the uncertainty set $\boldsymbol{\Delta}$, and it is an infinitely LMI-constrained problem. In many cases, this problem can be converted into a finite number of LMI constraints, and the condition will hold over the entire uncertainty set $\boldsymbol{\Delta}$ if they hold at vertices [6,7]. However, a major computational issue is that the number of LMIs may be excessively large and beyond the capabilities of existing LMI solvers.

To overcome this drawback, we follow a probabilistic approach to solve robust LMIs. Denote the maximum eigenvalue of the symmetric matrix $F(x, \Delta)$ as $\bar{\lambda}[F(x, \Delta)]$. Note that $\bar{\lambda}[F(x, \Delta)] < 0$ is a convex function in x, which is equivalent to $F(x, \Delta) < 0$. Therefore, the feasibility problem for the robust LMI (1) can be expressed as

$$\text{Find} \qquad x \in \boldsymbol{R}^n \quad \text{such that} \quad \bar{\lambda}[F(x, \Delta)] < 0 \quad \text{for all } \Delta \in \boldsymbol{\Delta} \tag{2}$$

Denote the solution set of the robust LMI problem (2) as

$$\mathcal{S} = \{x \in \boldsymbol{R}^n\colon \bar{\lambda}[F(x, \Delta)] < 0, \forall \Delta \in \boldsymbol{\Delta}\} \tag{3}$$

For a given probability distribution f_Δ, a probabilistic solution to (2) is an $x \in \boldsymbol{R}^n$ that satisfies $\text{Prob}\{\Delta \in \boldsymbol{\Delta}\colon \bar{\lambda}[F(x, \Delta)] < 0\} > \epsilon$, where $\epsilon < 1$ is a prespecified positive number denoting the expected probability of robust feasibility.

To apply the ellipsoid algorithm with a stopping rule, we need to choose an initial ellipsoid

$$E^{(0)} = \{x \in \boldsymbol{R}^n\colon (x - x^{(0)})^T (Q^{(0)})^{-1} (x - x^{(0)}) \le 1\} \tag{4}$$

where $x^{(0)} \in \boldsymbol{R}^n$ is the center of the ellipsoid, and $Q^{(0)}$ is a positive definite matrix. For the ellipsoid algorithm proposed by Kanev, the solution set $\mathcal{S}$ should be contained in the initial ellipsoid, and the problem of finding such an ellipsoid was discussed in detail [9]. This restriction is slightly relaxed through introducing the stopping rule.

There are two counters k and l in iterations. k denotes the number of iterations, and at each iteration a random sample $\Delta^{(k)}$ consistent with the probability distribution f_Δ is generated. l denotes the number of updates. At step k, if the inequality $\bar{\lambda}[F(x^{(k)}, \Delta^{(k)})] < 0$ is not satisfied, then the ellipsoid $E^{(k)}$ is updated.

Given an initial ellipsoid $E^{(0)}$, the maximum number of updates is defined as

$$\bar{l} = \left\lceil 2(n+1) \ln \frac{\text{vol} E^{(0)}}{\mu} \right\rceil \tag{5}$$

where the role of the operator $\lceil \cdot \rceil$ is to obtain the minimum integer greater than or equal to the real number inside. $\text{vol}E^{(0)}$ is the volume of the ellipsoid $E^{(0)}$ in $\boldsymbol{R}^n$, and it depends only on the determinant of $Q^{(0)}$ and the dimension of the variable space [17]. The parameter μ is chosen as a small positive number, which represents the volume of the final ellipsoid when the iteration is terminated compulsively. That is, if the volume of the ellipsoid sequence decreases to μ, and still no probabilistic feasible solution can be found, then the algorithm stops with no output. The detailed algorithm is stated as follows.

Algorithm 1 (*Ellipsoid Algorithm with Stopping Rule*):

1) Define parameters $0 < \epsilon < 1$, $\mu > 0$, $0 < \delta < 1$, and set $k = 0$, $l = 0$. Form an initial ellipsoid $E^{(0)}$ (4) with $x^{(0)} \in \boldsymbol{R}^n$, $Q^{(0)} \in \boldsymbol{R}^{n\times n}$, and $Q^{(0)} = (Q^{(0)})^T > 0$.

2) At step k, generate a random sample $\Delta^{(k)}$ according to the given probability distribution function f_Δ.

3) Check $\lambda^{(k)} = \bar{\lambda}[F(x^{(k)}, \Delta^{(k)})]$ and form the ellipsoid

$$E^{(k+1)} = \{x \in \boldsymbol{R}^n\colon (x - x^{(k+1)})^T (Q^{(k+1)})^{-1} (x - x^{(k+1)}) \le 1\} \tag{6}$$

If $\lambda^{(k)} < 0$, set $x^{(k+1)} = x^{(k)}$, $Q^{(k+1)} = Q^{(k)}$, and keep l unchanged.

If $\lambda^{(k)} \ge 0$, update the ellipsoid with

$$x^{(k+1)} = x^{(k)} - \frac{Q^{(k)} g^{(k)}}{(n+1)\sqrt{(g^{(k)})^T Q^{(k)} g^{(k)}}} \tag{7}$$

$$Q^{(k+1)} = \frac{n^2}{n^2 - 1}\left[Q^{(k)} - \frac{2Q^{(k)} g^{(k)} (g^{(k)})^T Q^{(k)}}{(n+1)(g^{(k)})^T Q^{(k)} g^{(k)}}\right] \tag{8}$$

and set $l = l + 1$. $g^{(k)}$ is a subgradient of $\bar{\lambda}[F(x^{(k)}, \Delta^{(k)})]$ and is given by

$$g^{(k)} = \begin{bmatrix} v^T F_1(\Delta^{(k)}) v \\ v^T F_2(\Delta^{(k)}) v \\ \vdots \\ v^T F_n(\Delta^{(k)}) v \end{bmatrix} \tag{9}$$

where v is any normalized eigenvector corresponding to the maximum eigenvalue $\bar{\lambda}[F(x^{(k)}, \Delta^{(k)})]$.

4) Check the stopping rules.

Case 1: Define

$$\kappa(l) = \left\lceil \frac{\ln([\pi^2 (l+1)^2]/6\delta)}{\ln(1/\epsilon)} \right\rceil \tag{10}$$

If $\lambda^{(k)}, \lambda^{(k-1)}, \ldots, \lambda^{(k-\kappa(l)+1)}$ are all well-defined and are all negative, stop and give $x^{(k+1)}$ as an output.

Case 2: If $l = \bar{l}$, stop with no output.

5) Set $k = k + 1$, and go back to step 2.

The proof of the algorithm can be found in [10]. Specifically, when the ellipsoid center does not change, i.e., the maximum eigenvalue is negative, for consecutive $\kappa(l)$ steps, the algorithm stops at case 1. The corresponding output x is a probabilistic solution with high confidence, and it means that x fails to satisfy $\text{Prob}\{\Delta \in \boldsymbol{\Delta}\colon F(x, \Delta) < 0\} > \epsilon$ with probability less than or equal to δ, which is usually chosen close enough to zero. When the algorithm stops at case 2, the volume of the set $E^{(0)} \cap \mathcal{S}$ is less than or equal to μ. It means that the choice of the initial ellipsoid $E^{(0)}$ is not good, or the solution set $\mathcal{S}$ is too small. If without considering the computational effort, however, we can always choose an initial ellipsoid of the volume big enough to intersect with the solution set when it does exist.

III. Robust LPV Control Using the Ellipsoid Algorithm

Consider an uncertain LPV system governed by

$$\begin{bmatrix} \dot{x}_p \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho, \Delta) & B_1(\rho, \Delta) & B_2(\rho, \Delta) \\ C_1(\rho, \Delta) & 0 & D_{12}(\rho, \Delta) \\ C_2(\rho, \Delta) & D_{21}(\rho, \Delta) & 0 \end{bmatrix} \begin{bmatrix} x_p \\ d \\ u \end{bmatrix} \tag{11}$$

where $x_p \in \boldsymbol{R}^{n_x}$ is the state, $e \in \boldsymbol{R}^{n_e}$ is the controlled output, $d \in \boldsymbol{R}^{n_d}$ is the disturbance, $u \in \boldsymbol{R}^{n_u}$ is the control input, and $y \in \boldsymbol{R}^{n_y}$ is the measurement output. All of the state-space data are continuous functions of the vector-valued parameter ρ, which is assumed to evolve continuously over time and its range is limited to a compact subset $\mathcal{P} \subset \boldsymbol{R}^s$.

The robust LPV controller is in the form of

$$\begin{bmatrix} \dot{x}_k \\ u \end{bmatrix} = \begin{bmatrix} A_k(\rho) & B_k(\rho) \\ C_k(\rho) & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y \end{bmatrix} \tag{12}$$

where $x_k \in \boldsymbol{R}^{n_k}$ is the state of the controller. The LPV controller gains A_k, B_k, and C_k can be determined using the following procedure.

Step 1: Synthesize robust Lyapunov matrices $R = R^T \in \boldsymbol{R}^{n_x \times n_x}$, $S = S^T \in \boldsymbol{R}^{n_x \times n_x}$ by solving the following set of LMIs for any $\rho \in \mathcal{P}$ and any $\Delta \in \boldsymbol{\Delta}$.

$$\mathcal{N}_R^T(\rho,\Delta)\begin{bmatrix} RA^T(\rho,\Delta)+(*) & * & * \\ C_1(\rho,\Delta)R & -\gamma I_{n_e} & * \\ B_1^T(\rho,\Delta) & 0 & -\gamma I_{n_d}\end{bmatrix}\mathcal{N}_R(\rho,\Delta) < 0 \tag{13}$$

$$\mathcal{N}_S^T(\rho,\Delta)\begin{bmatrix} A^T(\rho,\Delta)S+(*) & * & * \\ B_1^T(\rho,\Delta)S & -\gamma I_{n_d} & * \\ C_1(\rho,\Delta) & 0 & -\gamma I_{n_e}\end{bmatrix}\mathcal{N}_S(\rho,\Delta) < 0 \tag{14}$$

$$\begin{bmatrix} R & I_{n_x} \\ I_{n_x} & S \end{bmatrix} \geq 0 \tag{15}$$

where * means symmetric, and $\mathcal{N}_R(\rho,\Delta)$ and $\mathcal{N}_S(\rho,\Delta)$ denotes bases of null spaces of

$$\begin{bmatrix} B_2^T(\rho,\Delta) & D_{12}^T(\rho,\Delta) & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} C_2(\rho,\Delta) & D_{21}(\rho,\Delta) & 0 \end{bmatrix}$$

respectively. The robust solution guarantees nominal stability and nominal induced $\mathcal{L}_2$ gain performance $\frac{\|e\|_2}{\|d\|_2} < \gamma$.

Step 2: Construct the robust LPV controller based on nominal state-space matrices.

$$A_k(\rho) = -N^{-1}\Big\{A^T(\rho) + S[A(\rho) + B_2(\rho)K(\rho) + L(\rho)C_2(\rho)]R + \frac{1}{\gamma}S[B_1(\rho) + L(\rho)D_{21}(\rho)]B_1^T(\rho) + \frac{1}{\gamma}C_1^T(\rho)[C_1(\rho) + D_{12}(\rho)K(\rho)]R\Big\}M^{-T} \tag{16}$$

$$B_k(\rho) = N^{-1}SL(\rho) \tag{17}$$

$$C_k(\rho) = K(\rho)RM^{-T} \tag{18}$$

where matrices M and N are computed via singular value decomposition such that $MN^T = I - RS$, and matrix functions $K(\rho)$ and $L(\rho)$ are defined as

$$K(\rho) = -[D_{12}^T(\rho)D_{12}(\rho)]^{-1}[\gamma B_2^T(\rho)R^{-1} + D_{12}^T(\rho)C_1(\rho)]$$
$$L(\rho) = -[\gamma S^{-1}C_2^T(\rho) + B_1(\rho)D_{21}^T(\rho)][D_{21}(\rho)D_{21}^T(\rho)]^{-1}$$

Constraints (13–15) arise naturally from the standard LPV control [14,15] with the uncertainty present in data matrices. Note that (13–15) is based on single quadratic Lyapunov functions, i.e., variables R and S are constant matrices and this allows the parameter to vary arbitrarily fast. To reduce the conservativeness, parameter-dependent Lyapunov functions [16] can be used, where R and S are parameter-dependent matrix functions and the parameter variation rate is constrained. The presented randomized algorithm is still applicable but with increasing computational burden.

Theoretically, synthesis conditions (13–15) must hold for any value of the scheduling parameter $\rho \in \mathcal{P}$ and any uncertainty $\Delta \in \boldsymbol{\Delta}$. In this note, the parameter-dependency is dealt with using the "gridding" approach. Assume that m gridding points are selected, and they are dense enough to represent the dynamics over the entire parameter space $\mathcal{P}$. Then for any uncertainty $\Delta \in \boldsymbol{\Delta}$, the set of inequalities (13–15) should hold for $\rho = \rho_j$ with $j = 1, 2, \ldots, m$. Because the inequality (15) is independent of the parameter ρ for the case of single quadratic Lyapunov function, the LMI problem (13–15) can be expressed as $2m + 1$ robust LMIs, which can be solved using the proposed ellipsoid algorithm. For simplicity, we aim to find a feasible solution of R and S for a given performance level $\gamma > 0$. The original optimization problem can be solved using the bisection method [10].

To apply Algorithm 1, write the elements of symmetric matrix variables $R, S \in \boldsymbol{R}^{n_x \times n_x}$ into a vector variable $x \in \boldsymbol{R}^n$. Note that each matrix has $(n_x^2 + n_x)/2$ independent elements, and therefore $n = n_x^2 + n_x$. Denote the LMI (15) as $Y_0(x,\Delta)$. At each gridding point $\rho_j \in \mathcal{P}$, denote LMIs (13) and (14) as $Y_{j1}(x,\Delta)$, $Y_{j2}(x,\Delta)$, and combine them as a single LMI $Y_j(x,\Delta) = \mathrm{diag}\{Y_{j1}(x,\Delta), Y_{j2}(x,\Delta)\}$. The problem is recast as finding $x \in \boldsymbol{R}^n$ that satisfies

$$F(x,\Delta) = \mathrm{diag}\{Y_0(x,\Delta), Y_1(x,\Delta), \ldots, Y_m(x,\Delta)\} < 0 \tag{19}$$

The main computational efforts are on calculating the maximum eigenvalue of $F(x,\Delta)$ in (19) and its subgradient g. Note that the size of $F(x,\Delta)$ mainly depends on the order of the open-loop LPV system and the number of gridding points. For robust LPV control problem, $F(x,\Delta)$ is usually a matrix of large dimension, which may cause the ill-conditioning issue when computing its eigenvalues and eigenvectors. Because of the special block diagonal structure of $F(x,\Delta)$, the computation is based on the submatrix $Y_j(x,\Delta)$ instead of the augmented matrix $F(x,\Delta)$, and the possibility of numerical ill-conditioning can be minimized.

Remark 1 (*Computation of Maximum Eigenvalue*):

1) Compute the maximum eigenvalue $\bar{\lambda}_0$ and the corresponding eigenvector v_0 of the block $Y_0(x,\Delta)$. Set $\bar{\lambda} = \bar{\lambda}_0$ and $v = v_0$.

2) For $j = 1$ and $\rho = \rho_j$. Compute eigenvalues Λ_i and eigenvectors X_i of each block in $Y_j(x,\Delta)$. Find the maximum eigenvalue $\bar{\lambda}_j$ of the augmented eigenvalues $\Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2\}$, and determine the corresponding eigenvector v_j in the augmented eigenvectors $X = \mathrm{diag}\{X_1, X_2\}$. If $\bar{\lambda}_j > \bar{\lambda}$, set $\bar{\lambda} = \bar{\lambda}_j$ and $v = v_j$. Otherwise, keep $\bar{\lambda}$ and v unchanged.

3) Repeat the step 2 for $j = 2, 3, \ldots, m$.

Assume that the final values of $\bar{\lambda}$ and v are from the jth block $Y_j(x,\Delta)$. It is obvious that $\bar{\lambda}$ is the maximum eigenvalue of $F(x,\Delta)$. However, v is not the eigenvector of $F(x,\Delta)$, because its size is consistent with the submatrix $Y_j(x,\Delta)$, not $F(x,\Delta)$. Again, due to the block diagonal structure of $F(x,\Delta)$, v can be augmented as the eigenvector of $F(x,\Delta)$ with all elements as zero except those corresponding to the block $Y_j(x,\Delta)$. Based on the definition of the robust LMI in (1) and the given subgradient in (9), only the jth block $Y_j(x,\Delta)$ is involved when computing the subgradient of $F(x,\Delta)$.

Remark 2 (*Computation of Subgradient*):

1) For $i = 1$, set the ith element of x as 1, and others as 0. Rewrite the vector x as matrices R and S.

2) Compute the ith element of subgradient g as $v^T Y_j(\Delta) v$ at randomly selected Δ, where $Y_j(\Delta) = \mathrm{diag}\{Y_{j1}(\Delta), Y_{j2}(\Delta)\}$ if $j \neq 0$, or $Y_0(\Delta)$ if $j = 0$ with

$$Y_{j1}(\Delta) = \mathcal{N}_R^T(\rho_j,\Delta)\begin{bmatrix} RA^T(\rho_j,\Delta)+(*) & * & * \\ C_1(\rho_j,\Delta)R & 0 & * \\ 0 & 0 & 0\end{bmatrix}\mathcal{N}_R(\rho_j,\Delta)$$

$$Y_{j2}(\Delta) = \mathcal{N}_S^T(\rho_j,\Delta)\begin{bmatrix} A^T(\rho_j,\Delta)S+(*) & * & * \\ B_1^T(\rho_j,\Delta)S & 0 & * \\ 0 & 0 & 0\end{bmatrix}\mathcal{N}_S(\rho_j,\Delta)$$

$$Y_0(\Delta) = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$$

3) Repeat the steps 1–2 for $i = 2, 3, \ldots, n$.

To increase the chance of update and reduce the computation, Remark 1 can be further extended. For example, as soon as j is founded such that $\bar{\lambda}_j > 0$, the ellipsoid can be updated with the corresponding eigenvector v_j. However, the stopping rule needs to be modified accordingly.

IV. Application to Flight Control

In this section, the probabilistic LPV control design method is applied to flight dynamics. The system to be controlled is the longitudinal F-16 aircraft model based on NASA Langley Research Center (LaRC) wind tunnel tests [18]. The details of nonlinear modeling and simulations of the F-16 aircraft are presented in the NASA technical report [19].

A. Aircraft Model

In the full nonlinear longitudinal model, the states include true airspeed V, angle of attack α, pitch rate q, pitch angle θ, and actual engine power level pow (dimensionless). The control inputs are throttle position δ_{th} (dimensionless) and elevator angle δ_e. In addition, V, q, and flight path angle γ defined as $\theta - \alpha$ are selected as outputs. The flight envelope of interest covers the regime of $160 \leq V \leq 200$ ft/s and $20 \leq \alpha \leq 45$ deg. These two variables are used as scheduling parameters in the LPV modeling of F-16 longitudinal dynamics. The Jacobian linearization approach is used at 10 selected trim conditions to transform the nonlinear model of the system to an LPV model. By slight abuse of notations, the preceding state variables also represent perturbations from their equilibrium states when linearization is considered. This portion of flight envelope is chosen because the moderately high angle of attack causes large aerodynamic uncertainty, which is a major concern in this research.

The aerodynamic data are derived from low-speed static and dynamic wind tunnel tests conducted with subscaled models of the F-16 aircraft and provided in tabular form [18]. For simplicity, the approximate aerodynamic data in [20] are used, which reduces the size of the data from 50 lookup tables down to 10. There are only four lookup tables involved in the longitudinal model to provide aerodynamic stability derivatives C_{xq} and C_{zq}, aerodynamic force coefficients C_x and C_z, pitching moment derivative C_{mq}, and pitching moment coefficient C_m. As an initial study, we only consider the case that the coefficient C_{mq} is uncertain. The nominal value of C_{mq} is the function of the angle of attack. Assume that C_{mq} vary $\pm 100\%$ around its nominal values in the interested angle of attack region.

The nominal LPV model of F-16 is first derived, and then considering the uncertainty in C_{mq}, the uncertain LPV system is written as

$$\dot{x}_f = A_f(\rho, \Delta)x_f + B_f(\rho)u_f \qquad y_f = C_f x_f$$

where

$$x_f = \begin{bmatrix} V & \alpha & q & \theta & \text{pow} \end{bmatrix}^T$$

$$u_f = \begin{bmatrix} \delta_{\text{th}} & \delta_e \end{bmatrix}^T$$

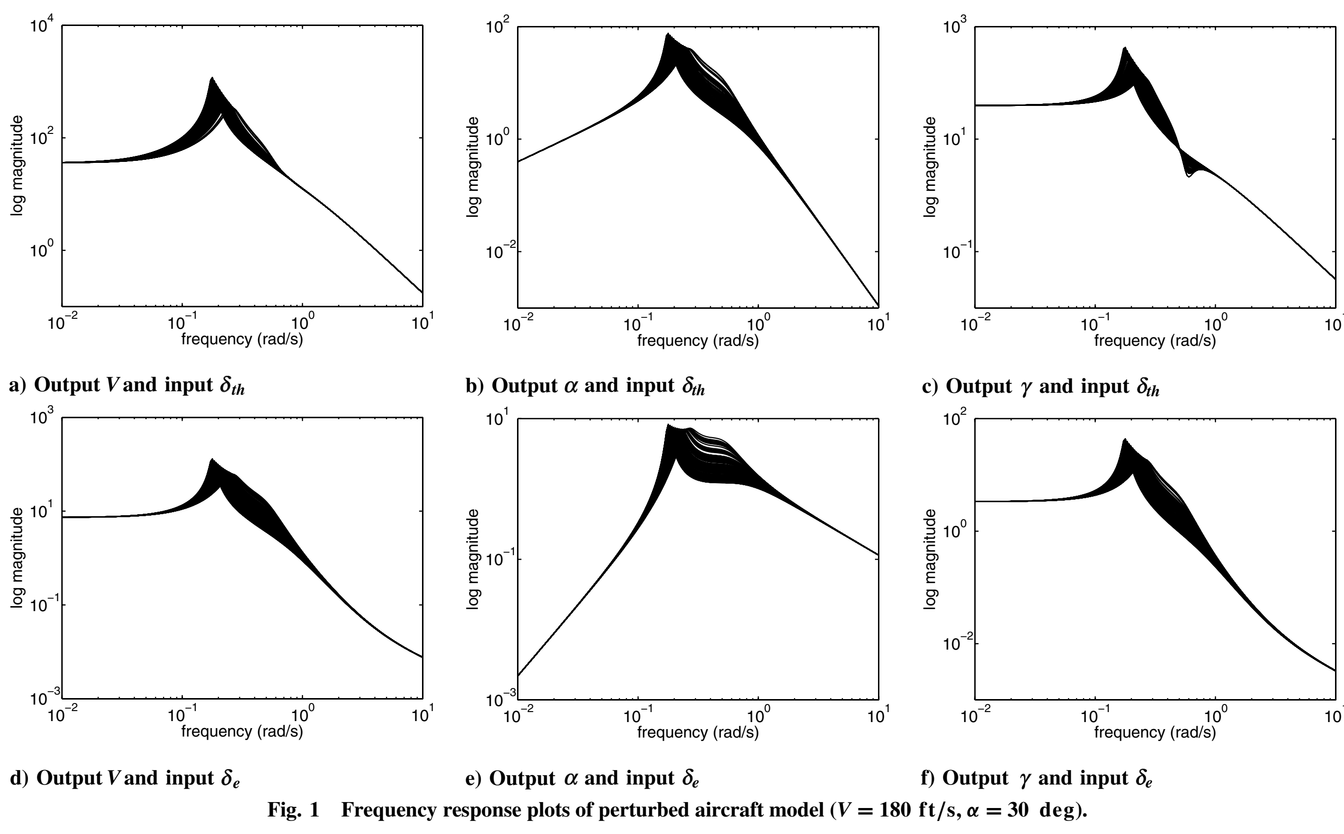
and

$$y_f = \begin{bmatrix} V & q & \gamma \end{bmatrix}^T$$

Through changing the value of C_{mq}, we found that the uncertainty in C_{mq} only causes the variation of (3, 3) element in the matrix A_f. Corresponding to $\pm 100\%$ variations of C_{mq}, the uncertain element (3, 3) of the matrix A_f will change $\pm 80\%$ around its nominal value. To see the effect of the aerodynamic coefficient C_{mq} on the flight dynamics, Fig. 1 shows the frequency responses of the aircraft model trimmed at $V = 180$ ft/s and $\alpha = 30$ deg, where 100 uncertainty samples are generated randomly. C_{mq} is observed to be one of the most influential coefficients affecting the system dynamics.

B. Controller Design

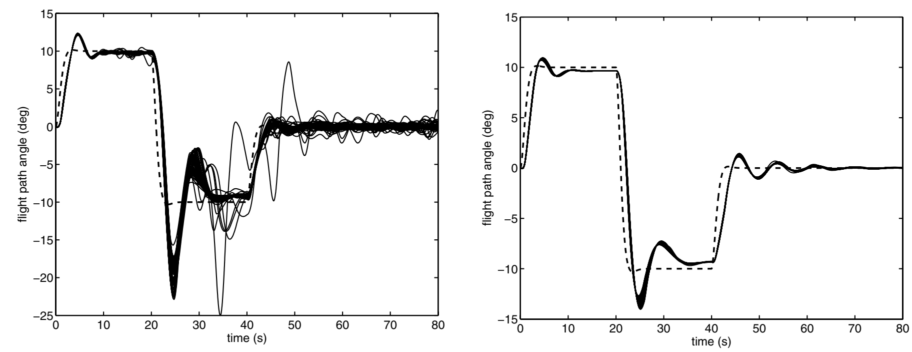
The control objective is to robustly track the flight path angle command under the uncertainty C_{mq}, and it is conveniently formulated as a model-following problem [21]. The detailed block diagram of the weighted system interconnection for synthesizing the LPV controller is presented in [21]. The state-space model of the open-loop weighted system is written in the form of (11), and only the state-space matrix A depends on the uncertainty Δ for this problem. The open-loop plant has the order of 10, and this leads to the LMI variable x to be a column vector with 110 elements. Without



a) Output V and input δ_{th} b) Output α and input δ_{th} c) Output γ and input δ_{th}

d) Output V and input δ_e e) Output α and input δ_e f) Output γ and input δ_e

Fig. 1 Frequency response plots of perturbed aircraft model ($V = 180$ ft/s, $\alpha = 30$ deg).



a) Nominal LPV controller

b) Robust LPV controller

Fig. 2 Nonlinear doublet response.

considering the aerodynamic uncertainty, R, S and γ in (13–15) were solved first, and the optimal performance level $\gamma = 48.47$ if the feasibility radius in the LMI solver is set as 10^4.

The robust LMIs (13–15) are then considered as a feasibility problem, and assume that the performance level $\gamma = 60$ by allowing for some gap above its nominal value. The nominal solutions R and S are converted to a column vector, which is taken as the initial value $x^{(0)}$ and is also the center of the initial ellipsoid $E^{(0)}$. The matrix $Q^{(0)}$, which determines the volume of $E^{(0)}$, is chosen as a diagonal matrix with the same number on the diagonal. Without considering computational effort, this number can be big enough to guarantee that there is an intersection between the initial ellipsoid and the solution set if it does exist. In this case, the volume of the initial ellipsoid is about 5.2832×10^{116}.

We set $\mu = 52.832$, $\epsilon = 0.9$, and $\delta = 0.1$ to execute the ellipsoid Algorithm 1, and the uniform probability distribution of uncertainty C_{mq} is assumed to randomly generate the uncertainty samples. With these parameter settings, the maximum number of updates is 58,785. In fact, after 8074 iterations and 7605 updates, the algorithm stopped and gave a probability solution x. It indicates that the performance level γ may be further improved. The volume of the ellipsoid at stop is about 7.7900×10^{101}.

C. Simulation Results

A Monte Carlo simulation was performed to estimate the empirical probability of robust feasibility. Ten thousand random samples of the uncertainty are generated based on the chosen uniform distribution function. For the nominal solution $x^{(0)}$, $\mathrm{Prob}\{\Delta \in \mathbf{\Delta}\colon F(x^{(0)}, \Delta) < 0\}$ is only 15.27%, whereas $\mathrm{Prob}\{\Delta \in \mathbf{\Delta}\colon F(x, \Delta) < 0\}$ is as high as 99.97% for the robust solution x.

To compare the performance of the closed-loop system, the nominal and robust controllers, constructed from $x^{(0)}$ and x, respectively, were tested at one designed flight condition, which is trimmed at $V = 180$ ft/s and $\alpha = 30$ deg. A flight path doublet input with magnitude ± 10 deg is used to demonstrate the performance of the robust controller. Both of nominal and robust controllers are LPV controllers, and their gains are calculated in real time by linear interpolation based on offline constructed lookup tables.

Figure 2a shows the response of the uncertain nonlinear longitudinal model controlled by the nominal LPV controller, and Fig. 2b is the response using the robust LPV controller. The dashed line in both plots represents the ideal response. Note that different from the Monte Carlo simulation, the aerodynamic coefficient C_{mq} is uncertain but fixed parameter in the LPV modeling during the nonlinear simulation. There are 100 uniform random samples of the uncertain aerodynamic coefficient C_{mq} for each case. It is observed that the tracking performance of the closed-loop system is greatly improved with smaller overshoot and less oscillation when employing the robust controller, and certainly more robust.

Other than uniform probability distribution case, the normally distributed uncertainty was also studied. Based on our observation, if the random algorithm and Monte Carlo simulations are both conducted using normal distributed uncertainty, high probability of robust feasibility and good tracking performance are often achieved. For instance, $\mathrm{Prob}\{\Delta \in \mathbf{\Delta}\colon F(x, \Delta) < 0\}$ is 99.05% for the case of normal distribution. Moreover, the solution computed from the uniform distribution can also achieve satisfactory results if the uncertainty is actually normally distributed. However, the results are slightly worse in the opposite way.

D. Discussion

Note that the preceding robust feasibility test is to check if robust LMIs (13–15) are satisfied by substituting the robust solution of matrices R and S. This set of LMIs only involve the data of the open-loop plant, and is derived from the following LMI by applying elimination lemma [22]:

$$\begin{bmatrix} A_{cl}^T(\rho,\Delta)P + PA_{cl}(\rho,\Delta) & PB_{cl}(\rho,\Delta) & C_{cl}^T(\rho,\Delta) \\ B_{cl}^T(\rho,\Delta)P & -\gamma I_{n_d} & D_{cl}^T(\rho,\Delta) \\ B_{cl}(\rho,\Delta) & D_{cl}(\rho,\Delta) & -\gamma I_{n_e} \end{bmatrix} < 0 \quad (20)$$

where the controller data are also involved through the state-space matrices of the closed-loop system. Generally, the controller depends on the parameter ρ and the uncertainty Δ as well. To come up with an implementable controller, we assumed that the controller only depends on ρ, and the controller matrices are constructed based on the nominal plant. To verify this assumption, another Monte Carlo simulation was performed using the LMI (20), where the closed-loop state-space matrices A_{cl}, B_{cl}, C_{cl}, and D_{cl} are derived following (11) and (12). P is the matrix associated with the Lyapunov function of the closed-loop system, and one of its valid choice is

$$P = \begin{bmatrix} S & R^{-1} - S \\ R^{-1} - S & S - R^{-1} \end{bmatrix}$$

The robust LPV controller is constructed using formula (16–18), where matrices R and S are converted from the vector x. If using the same distribution functions in both design and simulation, the probability of robust feasibility is about 85.31% for the uniform distribution case. The number is a little bit depressing; however, this might be the best possible with currently available robust synthesis techniques. Because the LMI (20) holds with fair confidence, the assumption that the controller is independent of uncertainty is practically acceptable.

V. Conclusions

The uncertainty in the aerodynamic modeling at high angle of attack presents significant challenges to control designers. An approach that is recently gaining popularity is to use randomized algorithms for robustness analysis and controller synthesis. This paper explored the applicability of an ellipsoid-based randomized algorithm to design a probabilistic robust LPV controller for a tactical aircraft. With a stopping rule, this algorithm can always stop to either give a probabilistic solution or detect no solution in an approximated sense. A dynamic output feedback controller was designed, and it has been shown through the nonlinear simulation that the performance of the closed-loop system is quite robust for random sampled uncertainties. The presented probabilistic robust control of F-16 with one uncertain aerodynamic coefficient is a promising first step leading to extensive future research. Multiple uncertain aerodynamic coefficients will be considered together, and the way that the uncertainty enters into the state-space matrices will be studied further.

Acknowledgments

This research was supported by NASA Langley Research Center under Grant NAG-1-01119 through Fen Wu while the first author was studying at NCSU. The authors would like to thank Eugene A. Morelli from NASA Langley Research Center for providing the nonlinear F-16 MATLAB model, and Yasuaki Oishi from University of Tokyo for his helpful support during this work.

References

[1] Alcom, C. W., Croom, M. A., Francis, M. S., and Ross, H., "The X-31 Aircraft: Advances in Aircraft Agility and Performance," *Progress in Aerospace Sciences*, Vol. 32, No. 4, 1996, pp. 377–413.

[2] Chiang, R. Y., Safonov, M. G., Haiges, K., Madden, K., and Tekawy, J., "A Fixed $\mathcal{H}_\infty$ Controller for a Supermaneuverable Fighter Performing the Herbst Maneuver," *Automatica*, Vol. 29, No. 1, 1993, pp. 111–127.

[3] Reigelsperger, W. C., Hammett, K. D., and Banda, S. S., "Robust Control Law Design for Lateral-Directional Modes of an F-16/MATV Using μ-Synthesis and Dynamic Inversion," *International Journal of Robust and Nonlinear Control*, Vol. 7, No. 8, 1997, pp. 777–795.

[4] Calafiore, G., Dabbene, F., and Tempo, R., "Randomized Algorithms in Robust Control," *Proceedings of 42nd IEEE Conference on Decision and Control*, IEEE, Piscataway, NJ, 2003, pp. 1908–1913.

[5] Vidyasagar, M., "Randomized Algorithms for Robust Controller Synthesis Using Statistical Learning Theory," *Automatica*, Vol. 37, No. 10, 2001, pp. 1515–1528.

[6] Calafiore, G., and Polyak, B. T., "Stochastic Algorithms for Exact and Approximate Feasibility of Robust LMIs," *IEEE Transactions on Automatic Control*, Vol. 46, No. 11, 2001, pp. 1755–1759.

[7] Polyak, B. T., and Tempo, R., "Probabilistic Robust Design with Linear Quadratic Regulators," *Systems & Control Letters*, Vol. 43, No. 5, 2001, pp. 343–353.

[8] Fujisaki, Y., Dabbene, F., and Tempo, R., "Probabilistic Design of LPV Control Systems," *Automatica*, Vol. 39, No. 8, 2003, pp. 1323–1337.

[9] Kanev, S., Schutter, B. D., and Verhaegen, M., "An Ellipsoid Algorithm for Probabilistic Robust Controller Design," *Systems & Control Letters*, Vol. 49, No. 5, 2003, pp. 365–375.

[10] Oishi, Y., "Polynomial-Time Algorithms for Probabilistic Solutions of Parameter-Dependent Linear Matrix Inequalities," *Automatica* (to be published).

[11] Marrison, C. I., and Stengel, R. F., "Design of Robust Control Systems for a Hypersonic Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 58–63.

[12] Wang, Q., and Stengel, R. F., "Robust Nonlinear Flight Control of a High-Performance Aircraft," *IEEE Transactions on Control Systems Technology*, Vol. 13, No. 1, 2005, pp. 15–26.

[13] Lu, B., and Wu, F., "Probabilistic Robust Control Design for an F-16 Aircraft," AIAA Paper 2005-6080, Aug. 2005.

[14] Becker, G., and Packard, A., "Robust Performance of Linear Parametrically Varying Systems Using Parametrically-Dependent Linear Feedback," *Systems & Control Letters*, Vol. 23, No. 3, 1994, pp. 205–215.

[15] Apkarian, P., and Gahinet, P., "A Convex Characterization of Gain-Scheduled $\mathcal{H}_\infty$ Controllers," *IEEE Transactions on Automatic Control*, Vol. 40, No. 5, 1995, pp. 853–864.

[16] Wu, F., Yang, X. H., Packard, A., and Becker, G., "Induced $\mathcal{L}_2$ Norm Control for LPV Systems with Bounded Parameter Variation Rates," *International Journal of Robust and Nonlinear Control*, Vol. 6, Nos. 9–10, 1996, pp. 983–998.

[17] Grötschel, M., Lovász, L., and Schrijver, A., *Geometric Algorithms and Combinatorial Optimization*, 2nd ed., Springer, Berlin, 1988.

[18] Nguyen, L. T., Ogburn, M. E., Gillert, W. P., Kibler, K. S., Brown, P. W., and Deal, P. L., "Simulator Study of Stall/Post-Stall Characteristics of a Fighter Airplane with Relaxed Longitudinal Static Stability," NASA Technical Rept. 1538, 1979.

[19] Garza, F. R., and Morellli, E. A., "A Collection of Nonlinear Aircraft Simulations in MATLAB," NASA Technical Rept. 212145, 2003.

[20] Stevens, B. L., and Lewis, F. L., *Aircraft Control and Simulation*, 2nd ed., John Wiley & Sons, Inc., New York, 2003.

[21] Lu, B., Wu, F., and Kim, S. W., "Linear Parameter-Varying Antiwindup Compensation for Enhanced Flight Control Performance," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 3, 2005, pp. 494–505.

[22] Gahinet, P., and Apkarian, P., "A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control," *International Journal of Robust and Nonlinear Control*, Vol. 4, No. 4, 1994, pp. 421–448.